A STUDY ON BK – ALGEBRAS

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Abstract

In this paper discuss and investigate a class of algebras which is related to several classes of algebras of interest such as BCIK-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, On B-algebra and quasi-groups, Quasi-groups and related system on algebras defined below demonstrates a rather interesting connection between B-algebras and groups. In this paper introduce combination of both BCIK/B-algebras define BK-algebra and its properties and also see the Commutative Derived BK-algebra.

***Keywords**: BCIK-algebra, B-algebra, BK-algebra, Commutative, Derived algebra.*

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INTRODUCTION

In 2021, S Rethina Kumar [1] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. They are two important classes of logical algebras, and have applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Also S Rethina Kumar introduced the notion of BCIK-algebra which is a generalization of a BCIK-algebra of a BCIK-algebra [1]. Several properties on BCIK-algebra are investigated in the papers [1,4]. But differently deal no negative meaning of information is suggested, now feel a need to deal with negative information. To do so, also feel a need to supply mathematical tool. To attain such object, introduced and use new function which is called negative valued function. The important achievement of this article is that one can deal with positive and negative information simultaneously by combining ideals in this article and already well known positive information. In[1-4] S Rethina Kumar, investigate a class of algebras which related to several classes of algebras of interest such as BCIK-algebra and which is seems to have rather nice properties without being excessively complicated otherwise. Furthermore, On B-algebra and quasi-groups, Quasi-groups and related system on algebras defined below demonstrates a rather interesting connection between B-algebras and groups. In this paper introduce combination of both BCIK/B-algebras define BK-algebra and its properties and also see the Commutative Derived BK-algebra.

BK-ALGEBRAS

A BK-algebra is a non-empty set X with a constant 0 and a binary "$*$" satisfying the following axioms hold for all $x, y, z \in X$:

(i) $0 \in X$,

(ii) $x * x = 0$,

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(iii) $x * 0 = x$,
(iv) $0 * x = 0$,
(v) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
(vi) $0 * (x * y) = (0 * x) * (0 * y)$,
(vii) $(x * y) * z = x * \big(z * (0 * y)\big)$.

Example 2.1. Let $X := \{0,1,2\}$ be a set with following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X;* ,0)$ is a BK-algebra.

Example 2.2. Let be the set of all real numbers except for a negative integer-n. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X;* ,0)$ is a BK-algebra.

Example 2.3. Let $X := \{0,1,2,3,4,5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X;* ,0)$ is a BK-algebra [1-5].

Example 2.4. Let $F\langle x,y,z\rangle$ be the free group on three elements. Define $u * v := vuv^{-2}$. Thus $u * u = e$ *and* $u * e = u$. Also $e * u = u^{-1}$. Now, given $a,b,c \in F\langle x,y,z\rangle$, let

$$\begin{aligned} w(a,b,c) &= ((a * b) * c)(a * (c * (e * b))^{-1} \\ &= (cbab^{-2}c^{-2})(b^{-1}cb^{2}a^{-1}cbcb^{2})^{-1} \\ &= cbab^{-2}c^{-2}b^{-2}c^{-1}b^{-1}c^{-1}ba^{-1}b^{-2}c^{-1}b. \end{aligned}$$

Let $N(*)$ be the normal subgroup of $F\langle x,y,z\rangle$ generated by the elements $w(a,b,c)$.

Let $G = F\langle x,y,z\rangle/N(*)$. On G define the operation " $\cdot$ " as usual and define $\big(uN(*)\big) * \big(vN(*)\big) := (u * v)N(*)$. It follows that

$\big(u\big(N(*)\big) * \big(uN(*)\big) = eN(*), \big(uN(*)\big) * \big(e * N(*)\big) = uN(*)$ and

$w\big(aN(*), bN(*), cN(*)\big) = w(a,b,c)N(*) = eN(*)$.

Hence $(G;*, eN(*))$ is a BK-algebra.

If we let $y := x$, then we have

$(x * x) * z = x * \big(z * (0 * x)\big)$.

If we let $z := x$, then we obtain also

$0 * x = x * \big(x * (0 * x)\big)$, using this it follows that

$0 = x * \big(0 * (0 * x)\big)$.

Let $X := (0,1,2\}$ be a set with the following left table:

$*$	0	1	2
0	0	1	0

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1	1	0	1
2	0	1	0

$*$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Let the set $X \coloneqq \{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

Then $(X;*,0)$ satisfies.

Lemma 2.5. If $(X;*,0)$ is a BK-algebra, then $y * z = y * (0 * (0 * z))$ for any $y, z \in X$.

Proof.

$$y * z = (y * z) * 0$$
$$= y * \big(0 * (0 * z)\big).$$

Lemma 2.6. If $(X;*,0)$ is a BK-algebra then $(x * y) * (0 * y) = x$ for any $x, y \in X$.

Proof. From $z = 0 * y$, we find that

$(x * y) * (0 * y) = x * \big((0 * y) * (0 * y)\big)$. Hence

$(x * y) * (0 * y) = x * 0$, so that its follows that

$(x * y) * (0 * y) = x$ as claimed.

Lemma 2.7. If $(X;*,0)$ is a BK-algebra then $x * z = y * z \; implies \; x = y$ for any $x, y, z \in X$.

Proof. If $x * z = y * z$, then $(x * z) * (0 * z) = (y * z) * (0 * z)$ and thus it follows that $x = y$.

Proposition 2.8. If $(X;*,0)$ is a BK-algebra, then

$x * (y * z) = \big(x * (0 * z)\big) * y for\ any\ x, y, z \in X$.

Proof. We obtain:

$$\big(x * (0 * z)\big) * y = x * \big(y * \big(0 * (0 * z)\big)\big)$$
$$= x * (y * z)).$$

Lemma 2.9. Let $(X;*,0)$ be a BK-algebra. Then for any $x, y \in X$,

(i) $x * y = 0 \; implies \; x = y$,

(ii) $0 * x = 0 * y \; implies \; x = y$,

(iii) $0 * (0 * x) = x$.

Proof.

(i) Since $x * y = 0$ implies $x * y = y * y$, it follows that $x = y$.

(ii) $0 * x = 0 * y, then\ 0 = x * x = (x * x) * 0 = x * \big(0 * (0 * x)\big) = x * \big(0 * (0 * y)\big) = (x * y) * 0 = x * y, and\ thus\ x = y.$

(iii) For any $x \in X$, we obtain $0 * x = (0 * x) * 0 = 0 * \big(0 * (0 * x)\big)$, it follows that $x = 0 * (0 * x)$ as claimed.

Note: Let $(X;*,0)$ be a BK-algebra and let $g \in X$. Define $g^n \coloneqq g^{n-1} * (0 * g)(n \geq 1)\ and$ $g^0 \coloneqq 0$. Note that $g^1 = g^0 * (0 * g) = 0 * (0 * g) = g$

Lemma 2.10. Let $(X;*,0)$ be a BK-algebra and let $g \in X$. Then $g^n * g^m = g^{n-m}$ where $n \geq m$.

Proof. If X is a BK-algebra, then note that it follows that

$$g^2 * g = \big(g^1 * (0 * g)\big) * g = \big(g * (0 * g)\big) * g = g\Big(g * \big(0 * (0 * g)\big)\Big) = g * (g * g) = g * 0 = g.$$

Assume that $g^{n+1} * g = g^n (n \geq 1)$. Then

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$g^{n+2} * g = (g^{n+1} * (0 * g)) * g$
$= g^{n+1} * \left(g * (0 * (0 * g))\right)$
$= g^{n+1} * 0.$
$= g^{n+1}.$

Assume $g^n * g^m = g^{n-m}$ where $n - m \geq 1$. Then
$g^n * g^{m+1} = (g^n * (g^m * (0 * g))$
$= (g^n * g) * g^m$
$= g^{n-1} * g^m$
$= g^{n-(m+1)},\ [Since\ n - m - 1 \geq 0]$

Proving the lemma.

Lemma 2.11. Let $(X;* ,0)$ be a BK-algebra and let $g \in X$. Then $g^m * g^n = 0 * g^{n-m}\ where\ n > m.$

Proof. If X is a BK-algebra then, we have $g * g^2 = g * \left(g^1 * (0 * g)\right) = (g * g) * g^1 = 0 * g.$

Assume that $g * g^n = g^{n-1}\ where\ (n \geq 1)$. Then
$g * g^{n+1} = g * \left(g^n * (0 * g)\right)$
$= (g * g) * g^n$
$= 0 * g^n.$

Assume that $g^m * g^n = g^{n-m}$ where $n - m \geq 1$. Then
$g^{m+1} * g^n = (g^m * (0 * g)) * g^n$
$= g^m * (g^n * g)$
$= g^m * g^{n-1}$
$= 0 * g^{n-m-1},$

Proving the lemma.

Theorem 2.12. Let $(X;* ,0)$ be a BK-algebra and let $g \in X$. Then

$$g^m * g^n = \begin{cases} g^{m-n}\ if\ m \geq n, \\ 0 * g^{n-m}\ otherwise. \end{cases}$$

Proposition 2.13. If $(X;* ,0)$ is a BK-algebra, then $(a * b) * b = a * b^2$ for any $a, b \in X$.

Proof. $(a * b) * b = a * \left(b * (0 * b)\right) = a * b^2.$

Proposition 2.14. If $(X;* ,0)$ is a BK-algebra, then $(0 * b) * (a * b) = 0 * a$ for any $a, b \in X$.

Proof. $(0 * b) * (a * b) = \left((0 * b) * (0 * b)\right) * a = 0 * a.$

COMMUTATIVITY

A BK-algebra $(X;* ,0)$ is said to be commutative if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$.

Proposition 3.1. If $(X;* ,0)$ is a commutative BK-algebra, then $(0 * x) * (0 * y) = y * x\ for\ any\ x, y \in X.$

Proof. Since X is commutative, $(0 * x) * (0 * y) = y * \left(0 * (0 * x)\right) = y * x.$

Theorem 3.2. If $(X;* ,0)$ is a commutative BK-algebra, then $(a * (a * b) = b$ for any $a, b \in X$.

Proof. If X is commutative, $a * (a * b) = \left(a * (0 * b)\right) * a = \left(b * (0 * a)\right) * a = b * (a * a) = b.$

Corollary 3.3. If $(X;* ,0)$ is a commutative BK-algebra, then the left cancellation law holds, i.e., $a * b = a * b'\ implies\ b = b'.$

Proof.$b = a * (a * b) = a * (a * b') = b'.$

Proposition 3.4. If $(X;* ,0)$ is a commutative BK-algebra, then $(0 * a) * (a * b) = b * a^2$ for any $a, b \in X.$

Proof. If X is a commutative BK-algebra, then
$(0 * a) * (a * b) = ((0 * a) * (0 * b) * a = (b * a) * a = b * a^2.$

DERIVED ALGEBRA AND BK-ALGEBRAS

Given algebras (i.e., groupoids, binary systems) $(X;*)$ *and* $(X;\circ)$, it is often argued that they are "essentially equivalent" when they are not, and even if it is perfectly clear how proceed from to the other and back again, it is also clear that knowledge of one "implies" knowledge of the other in a complete enough sense as to have the statement that they are "essentially equivalent" survive closer inspection.

We proceed the integers Z, we consider the system $(Z;+,0)$ as an abelian group with identity 0. If we consider the system $(Z;-,0)$, then we can reproduce $(Z;+,0)$ by "defining" $x+y \coloneqq x-(0-y)$, and observing that in the first case "0 is the unique element such that $x-0=x$ for all x, while in the second case "0 is the unique element such that $x+0=x$ for all x".

However, that is by no means all we might have said to identify 0 nor is it necessary what we need it say to identify 0 in this setting.

Let $(X;*,0)$ *and* $(X;\circ,0)$ be algebras. We denote $(X;*,0) \longrightarrow (X;\circ,0)$ if $x\circ y = x*(0*y)$, for all $x,y \in X$. The algebra $(X;\circ,0)$ is said to be derived from the algebra $(X;*,0)$. Let V be the set of all algebras defined on X and let $\Gamma_d(V)$ be the digraph whose vertices are V and whose described above.

Example 4.1. ([1-5]) If we define $x*y \coloneqq \max\{0, \frac{x(x-y)}{x+y}\}$ on X, then $(X;*,0)$ is a BK-algebra.

Proposition 4.2. The derived algebra $(X;*,0)$ from BK-algebra $(X;*,0)$ is a left zero semigroup.

Proof. Let $(X;*,0)$ be a BK-algebra and let $(X;*,0) \longrightarrow (X;\circ,0)$. Then $x\circ y = x*(0*y)$,

For any $x,y \in X$. Since $(X;*,0)$ is a BK-algebra, $x*(0*y) = x*0 = x, i.e., x\circ y = x$, proving that $(X;*,0)$ is a left zero semigroup.

Notice that such an arrow in $\Gamma_d(V)$ can always be constructed, but it is not true that a backward arrow always exists. For example, since every BCIK-algebra $(X;*,0)$ is a BK-algebra, we have $(X;*,0) \longrightarrow (X;\circ,0)$ where $(X;\circ,0)$ is a left zero semigroup. Assume that $(X;\circ,0) \longrightarrow (X;*,0)$, where $(X;*,0)$ is a non-trivial BCIK-algebra. Then $x*y = x\circ(0\circ y)$, for any $x,y \in X$. Since $(X;\circ)$ is a left zero semigroup, we have $x,y = x$ for any $,y \in X$, contradicting that $(X;*,0)$ is a BCIK-algebra.

The most interesting result in this context may be:

Theorem 4.3. Let $(X;*,0)$ be a BK-algebra. If $(X;*,0) \longrightarrow (X;\circ,0)$, i.e., if $x\circ y = x*(0*y)$, then $(X;\circ,0)$ is a group.

Proof. If $(X;*,0) \longrightarrow (X;\circ,0)$, then $x\circ y = x*(0*y)$, for any $x,y \in X$. By $0*(0*x) = x\, for\, any\, x \in X$, i.e., $x = 0\circ x$. Since $x\circ 0 = x*(0*0) = x*0 = x$, 0 acts like an identity element of X. Routine calculations show that $(X;\circ,0)$ is a group.

Proposition 4.4. The derived algebra from a group is that group itself.

Proof. Let $(X;*,0)$ be a group with identity 0. If $(X;*,0) \longrightarrow (X;\circ,0)$, then $x\circ y = x*(0*y) = x*y$, Since 0 is the identity, for any $x,y \in X$. This proves the proposition.

Thus, we can use the $\longrightarrow$ mechanism to proceed from the BK-algebra to the groups, but since groups happen to be sinks in this graph, we cannot use $\longrightarrow$ mechanism to return from groups to

BK-algebra. This does not mean that there are no other ways to do so, but it does argue for the observation that BK-algebras are not only "different", but in a deep sense "non-equivalent", and from the point of view of the digraph $\Gamma_d(V)$ the BK-algebra is seen to be a predecessor of the group.

Given a group $(X;\cdot,e)$, if we define $x*y \coloneqq x\cdot y^{-1}$, then $(X;*,0=e)$ is seen to be a BK-algebra, and furthermore, it also follows that $(X;*,0=e) \to (X;\cdot,e)$, since $x*(e*y) = x\cdot(e\cdot y^{-1})^{-1} = x\cdot(y^{-1})^{-1} = x\cdot y$.

The problem here is that there is not a formula involving only $(X;\cdot,e)$ which produces $x*y$, i.e., we have to introduce $(X;\cdot,-^{-1},e)$ as the type to describe a group to permit us to perform this task. In fact, we may use this observation as another piece of evidence that BK-algebra $(X;*,0)$ are not "equivalent" to groups $(X;*,0)$. If we introduce the mapping $x \to 0*x$ is not a new item which needs to be introduced, while in the case of groups it is.

The difficulty we are observing in the situation above is also visible in the case of "the subgroup test". If $(X;\cdot,e)$ is an infinite group, and if $\emptyset \neq S \subseteq X$, then if S is closed under multiplication it is not the case that S need a subgroup. Indeed, the rule is that if $x,y \in S$, then also $x\cdot y^{-1} \in S$. From what we have already seen, $x\cdot y^{-1}$ is precisely the element $x*y$ if $(X;*,e) \to (X;\cdot,e)\ in\ \Gamma_d(V)$. Thus we have the following "subgroup test" for BK-algebra: $\emptyset \neq S \subseteq X$ is a sub algebra of the BK-algebra $(X;*,0)$, precisely when $x,y \in S$ implies $x,y \in S$. Also, suppose $(X;*,0) \to (X;\cdot,e=0)\ in\ \Gamma_d(V)$ where it is given that $(X;\cdot,e)$ is a group. Then it is not immediately clear that $(X;*,0)$ must be a uniquely defined BK-algebra, even if we know that there is at least one BK-algebra with this property.

CONCLUSION

In this paper introduce BK-algebra (commutative) derived algebra and BK-algebra.

In our future study some useful properties of this a complete ideal in extended in various algebraic structure of Q-algebras, subtraction algebras, d-algebra and so forth.

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